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HEAT AND MASS TRANSFER

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WHY HEAT AND MASS TRANSFER

Heat transfer and mass transfer are kinetic processes that may occur and be studied separately or jointly. Studying them apart is simpler, but both processes are modelled by similar mathematical equations in the case of diffusion and convection (there is no mass-transfer similarity to heat radiation), and it is thus more efficient to consider them jointly. Besides, heat and mass transfer must be jointly considered in some cases like evaporative cooling and ablation.

The usual way to make the best of both approaches is to first consider heat transfer without mass transfer, and present at a later stage a briefing of similarities and differences between heat transfer and mass transfer, with some specific examples of mass transfer applications. Following that procedure, we forget for the

moment about mass transfer (dealt with separately under Mass Transfer), and concentrate on the simpler problem of heat transfer.

There are complex problems where heat and mass transfer processes are combined with chemical reactions, as in combustion; but many times the chemical process is so fast or so slow that it can be decoupled and considered apart, as in the important diffusion-controlled combustion problems of gas-fuel jets, and condensed fuels (drops and particles), which are covered under Combustion kinetics. Little is mentioned here about heat transfer in the micrometric range and below, or about biomedical heat transfer (see Human thermal comfort).

FUNDAMENTALS OF HEAT TRANSFER (WHAT IS IT)

Heat transfer is the flow of thermal energy driven by thermal non-equilibrium (i.e. the effect of a non-uniform temperature field), commonly measured as a heat flux (vector), i.e. the heat flow per unit time (and usually unit normal area) at a control surface.

The aim here is to understand heat transfer modelling, but the actual goal of most heat transfer (modelling) problems is to find the temperature field and heat fluxes in a material domain, given a previous knowledge of the subject (general partial differential equations, PDE), and a set of particular constraints: boundary conditions (BC), initial conditions (IC), distribution of sources or sinks (loads), etc. There are also many cases where the interest is just to know when the heat-transfer process finishes, and in a few other cases the goal is not in the direct problem (given the PDE+BC+IC, find the T-field) but on the inverse problem: given the T-field and some aspects of PDE+BC+IC, find some missing parameters (identification problem), e.g. finding the required dimensions or materials for a certain heat insulation or conduction goal.

Heat-transfer problems arise in many industrial and environmental processes, particularly in energy utilization, thermal processing, and thermal control. Energy cannot be created or destroyed, but so-common it is to use energy as synonymous of exergy, or the quality of energy, than it is commonly said that energy utilization is concerned with energy generation from primary sources (e.g. fossil fuels, solar), to end-user energy consumption (e.g. electricity and fuel consumption), through all possible intermediate steps of energy valorisation, energy transportation, energy storage, and energy conversion processes. The purpose of thermal processing is to force a temperature change in the system that enables or disables some material transformation (e.g. food pasteurisation, cooking, steel tempering or annealing). The purpose of thermal control is to regulate within fixed established bounds, or to control in time within a certain margin, the temperature of a system to secure its correct functioning.

As a model problem, consider the thermal problem of heating a thin metallic rod by grasping it at one end with our fingers for a while, until we withdraw our grip and let the rod cool down in air; we may want to predict the evolution of the temperature at one end, or the heat flow through it, or the rod conductivity needed to heat the opposite end to a given value. We may learn from this case study how difficult it is to model the heating by our fingers, the extent of finger contact, the thermal convection through the air, etc. By

the way, if this example seems irrelevant to engineering and science (nothing is irrelevant to science), consider its similarity with the heat gains and losses during any temperature measurement with a typical 'long' thermometer (from the old mercury-in-glass type. to the modern shrouded thermocouple probe). A more involved problem may be to find the temperature field and associated dimensional changes during machining or cutting a material, where the final dimensions depend on the time-history of the temperature field.

Everybody has been always exposed to heat transfer problems in normal life (putting on coats and avoiding winds in winter, wearing caps and looking for breezes in summer, adjusting cooking power, and so on), so that certain experience can be assumed. However, the aim of studying a discipline is to understand it in depth; e.g. to clearly distinguish thermal-conductivity effects from thermal-capacity effects, the relevance of thermal radiation near room temperatures, and to be able to make sound predictions. Typical heat-transfer devices like heat exchangers, condensers, boilers, solar collectors, heaters, furnaces, and so on, must be considered in a heat-transfer course, but the emphasis must be on basic heat-transfer models, which are universal, and not on the myriad of details of past and present equipment.

Heat transfer theory is based on thermodynamics, physical transport phenomena, physical and chemical energy dissipation phenomena, space-time modelling, additional mathematical modelling, and experimental tests.

Thermodynamics of heat transfer

Heat transfer is the relaxation process that tends to do away with temperature gradients (recall that $\nabla T \rightarrow 0$ in an isolated system), but systems are often kept out of equilibrium by imposed boundary conditions. Heat transfer tends to change the local state according to the energy balance, which for a closed system is:

$$\text{What is heat } (\equiv\text{heat flow})? \quad Q \equiv \Delta E - W \;\rightarrow\; Q = \Delta E|_{V,\text{non-dis}} = \Delta H|_{p,\text{non-dis}} \qquad (1)$$

i.e. heat, Q (i.e. the flow of thermal energy from the surroundings into the system, driven by thermal non-equilibrium, not related to work or to the flow of matter), equals the increase in stored energy, ΔE, minus the flow of work, W. For non-dissipative systems (i.e. without mechanical or electrical dissipation), heat equals the internal energy change if the process is at constant volume, or the enthalpy change if the process is at constant volume, both cases converging for a perfect substance model (PSM, i.e. constant thermal capacity) to $Q = mc\Delta T$. However, it is worth to keep in mind that:

- Heat is the flow of thermal energy driven by thermal non-equilibrium, so that 'heat flow' is a redundancy (i.e. a pleonasm, and the same for ‘work flow’). Heat must not be confused with stored thermal energy, and moving a hot object from one place to another must not be called heat transfer. But, in spite of all these remarks, it is common in normal parlance to say ‘heat flow’, to talk of ‘heat content’, etc.

- Heat is an energy flow, defined by (1) just for the case of mass-impervious systems (i.e. $Q \equiv W_{\text{adiab}} - W$). When there are simultaneous energy and mass flows, heat flow must be considered at a surface with no net mass flow.
- Heat input to a system, may not necessarily cause a temperature increase. In absence of work, a heat input always increases internal energy ($Q=\Delta E$ for $W=0$ in (1)), but this increment may be 'sensible' (i.e. noticeable as a temperature increase), or 'latent' (e.g. causing a phase change or other endothermic reaction at constant temperature).
- A temperature increase in a closed system is not necessarily due to a heat input; it can be due to a work input (e.g. $\Delta E = mc\Delta T = W$ for $Q=0$ in (1)), either with dissipation (e.g. internal stirring), or without (isentropic compression).
- The First Law (1) shows that, for a steady state without work exchange, the heat loss by a system must pass integrally to another system, i.e. for the interface, $Q \equiv Q_{\text{net}} = Q_{\text{in}} - Q_{\text{out}} = 0$ for $\Delta E = W = 0$.
- The Second Law teaches that heat always flows from the hotter system towards the colder one. Even when we want to extract heat from a cold system like in refrigeration, we must procure a colder working substance for heat to flow down the temperature gradient to the working fluid (later to be compressed to a higher temperature than that of the heat sink, to finally dispose of the thermal energy again by letting heat to flow down the temperature gradient to the ambient).

In Thermodynamics, sometimes one refers to heat in an isothermal process, but this is a formal limit for small gradients and large periods. Here, in Heat Transfer, the interest is not in heat flow, Q, but on heat-flow-rate, $\dot{Q} = \mathrm{d}Q/\mathrm{d}t$, that should be named just heat rate, because the 'flow' characteristic is inherent to the concept of heat, contrary for instance to the concept of mass, to which two possible 'speeds' can be ascribed: mass rate of change, and mass flow rate. Heat rate, thence, is energy flow rate at constant volume, or enthalpy flow rate at constant pressure:

$$\text{What is heat flux (}\equiv\text{heat flow rate)?} \quad \dot{Q} \equiv \frac{dQ}{dt} = mc\left.\frac{dT}{dt}\right|_{\text{PSM,non-dis}} \equiv KA\Delta T \qquad (2)$$

where the global heat transfer coefficient K (associated to a bounding area A and the average temperature jump ΔT between the system and the surroundings), is defined by (2); the inverse of K is named global heat resistance coefficient $M \equiv 1/K$. Notice that this is the recommended nomenclature under the SI, with $G=KA$ being the global transmittance and $R=1/G$ the global resistance, although U has been used a lot instead of K, and R instead of M.

In most heat-transfer problems, it is undesirable to ascribe a single average temperature to the system, and thus a local formulation must be used, defining the heat flow-rate density (or simply heat flux) as $\dot{q} \equiv \mathrm{d}\dot{Q}/\mathrm{d}A$. According to the corresponding physical transport phenomena explained below, heat flux can be related to temperature difference between the system and the environment in the classical three modes of conduction, convection, and radiation:

$$\text{What is heat flux density (≈heat flux)?} \quad \dot{q} = K\Delta T \begin{cases} \text{conduction} & \vec{\dot{q}} = -k\nabla T \\ \text{convection} & \dot{q} \equiv h\left(T - T_\infty\right) \\ \text{radiation} & \dot{q} = \varepsilon\sigma\left(T^4 - T_0^4\right) \end{cases} \quad (3)$$

These three heat-flux models can also be viewed as: heat transfer within materials (conduction), heat transfer within fluids (convection), and heat transfer through empty space (radiation).

Notice that heat (related to a path integral in a closed control volume in thermodynamics) has the positive sign when it enters the system, but heat flux, related to a control area, cannot be ascribed a definite sign until we select 'our side'. For heat conduction, (3) has a vector form, stating that heat flux is a vector field aligned with the temperature-gradient field, and having opposite sense. For convection and radiation, however, (3) has a scalar form, and, although a vector form can be forged multiplying by the unit normal vector to the surface, this commonly-used scalar form suggest that, in typical heat transfer problems, convection and radiation are only boundary conditions and not field equations as for conduction (when a heat-transfer problem requires solving field variables in a moving fluid, it is studied under Fluid Mechanics). Notice also that heat conduction involves field variables: a scalar field for T and a vector field for $\vec{\dot{q}}$, with the associated differential equations relating each other (because only short-range interactions are involved), which are partial differential equations because time and several spatial coordinates are related.

Another important point in (3) is the non-linear temperature-dependence of radiation, what forces to use absolute values for temperature in any equation with radiation effects. Conduction and convection problems are usually linear in temperature (if k and h are T-independent), and it is common practice working in degrees Celsius instead of absolute temperatures.

Finally notice that (1) and (2) correspond to the First Law (energy conservation), and (3) incorporates the Second-Law consequence of heat flowing downwards in the T-field (from hot to cold).

Physical transport phenomena

Heat flow is traditionally considered to take place in three different basic modes (sometimes superposed): conduction, convection, radiation.

- Conduction is the transport of thermal energy in solids and non-moving fluids due to short-range atomic interactions, supplemented with the free-electron flow in metals, modelled by the so-called Fourier's law (1822), $\vec{\dot{q}} = -k\nabla T$, where k is the so-called thermal conductivity coefficient (see below). Notice that Fourier's law has a local character (heat flux proportional to local temperature gradient, independent of the rest of the T-field), what naturally leads to differential equations. Notice also that Fourier's law implies an infinite speed of propagation for temperature gradients (thermal waves), which is nonsense; thermal conduction waves propagate at the speed of sound in the medium, as any other phenomena small perturbation. In crystalline solids, packets of quantised energy called phonons serve to explain thermal conduction (as photons do in electromagnetic radiation).

- Convection, in the restricted sense used in most Heat Transfer books, is the transport of thermal energy between a solid surface (at wall temperature T) and a moving fluid (at a far-enough temperature T_∞), modelled by a thermal convection coefficient h as in the second line of (3), named Newton's law (1701); in this sense, heat convection is just heat conduction at the fluid interface in a solid, whereas in the more general sense used in Fluid Mechanics, thermal convection is the combined energy transport and heat diffusion flux at every point in the fluid. Notice, however, that what goes along a hot-water insulated pipe is not heat and there is no heat-transfer involved; it is thermal energy being convected, without thermal gradients. Related to fluid flow, but through porous media is percolation; a special case concerns heat transfer in biological tissue by blood perfusion (i.e. the flow of blood by permeation through tissues: skin, muscle, fat, bone, and organs, from arteries to capillaries and veins); the cardiovascular system is the key system by which heat is distributed throughout the body, from body core to limbs and head.
- Radiation is the transport of thermal energy by far electromagnetic coupling, modelled from the basic black-body theory (fourth-power-law of thermal emission), $M_{bb}=\sigma T^4$, named Stefan-Boltzmann law (proposed by Jozef Steafan in 1879 and deduced by his student Ludwig Bolzmann in 1884), σ being a universal constant $\sigma=5.67\cdot10^{-8}$ W.m^{-2}.K^{-4}, modified for real surfaces by introducing the emissivity factor, ε ($0<\varepsilon<1$). Radiation is emitted as a result of the motion of electric charges in atoms and molecules (by thermal vibrations or external forces), and radiation is absorbed by matter increasing the atomic motion of electric particles. Notice that (3) only applies to radiation heat transfer when the surface absorptance to radiation coming from the environment at temperature T_0 is equal to surface emissivity at temperature T, what is usually not the case if both temperatures are of different orders of magnitude. If absorption at a surface is not total, part of the incident radiation may be reflected or transmitted behind (scattered, in general). Heat transfer by thermal radiation is not only important at high temperatures; even at room temperatures, $T\approx300$ K, the equivalent linear coefficient of heat transfer is $K=4\sigma T^3=6.1$ W/(m$^2\cdot$K), comparable to a typical natural convective coefficient in air of $h=10$ W/(m$^2\cdot$K).

The three heat-transfer modes above-mentioned, often appear at the same time on a thermal problem, but seldom with the same importance, what allows for simple one-mode analysis in many instances. A combined case that appears in many cases is the heat flow by convection from one fluid to another fluid separated by an intermediate solid wall (single, as in pipes, or double, as in modern window panes); in such cases, dealing with an overall heat-transfer coefficient, K, is very helpful (one has just to apply (2)); e.g. typical values for a modern house may be: $K=0.5$ W/(m$^2\cdot$K) for walls, $K=0.6$ W/(m$^2\cdot$K) for the roof, $K=0.7$ W/(m$^2\cdot$K) for the floor, and $K=3$ W/(m$^2\cdot$K) for a double-pane window.

Notice finally that isothermal surfaces are usually assumed in convection- and radiation- heat-transfer problems, and that the temperature field is only solved in heat conduction problems (except when big computation codes are used to solve the whole fluid mechanics problem, namely in CFD).

Thermal conductivity

The thermal conductivity, k, of a given isotropic material at given conditions, is the proportionality constant defined by Fourier's law, $\vec{q} = -k\nabla T$. For non-isotropic materials, k is no-longer a scalar magnitude but a tensor. Representative k-values are presented in Table 1.

Table 1. Representative thermal conductivities.

	k [W/(m·K)]	Comments
Order of magnitude for solids	10^2 (good conductors) 1 (bad conductors)	In metals, Lorentz's law (1881), $k/(\sigma T)$=constant, relates thermal conductivity, k, to electrical conductivity, σ, when electron-gas conduction is dominant; however, good electrical conductivity is not synonymous of good thermal conductivity: mercury only has $k\approx 9$ W/(m·K) being a good electrical conductor, and diamond has $k\approx 1000$ W/(m·K) being a good electrical insulator. Phonon theory indicates that electrical insulators have $k(T<<T_D)\propto T^3$, and $k(T>>T_D)\propto 1/T$, whereas pure metals have an electronic contribution (on top of that of phonons) of $k(T<<T_D)\propto T$, and $k(T>>T_D)\approx$constant. Some values for metal alloys are in Table 2 below. Ceramics and polymers are, in general poor conductors (see Solid data).
Aluminium	200	Very pure aluminium may reach k=237 W/(m·K) at 288 K, decreases to k=220 W/(m·K) at 800 K; going down, k=50 W/(m·K) at 100 K, increasing to a maximum of $k=25\cdot 10^3$ W/(m·K) at 10 K and then decreasing towards zero proportionally to T, with $k=4\cdot 10^3$ W/(m·K) at 1 K). Duralumin (4.4%Cu, 1%Mg, 0.75%Mn, 0.4%Si) has k=174 W/(m·K), increasing to k=188 W/(m·K) at 500 K.
Iron and steel	50	Pure iron (ferrite) has k=80 W/(m·K). Cast iron (96%Fe, 4%C) has k=40 W/(m·K). Mild-carbon steel with <0.4%C have k=52 W/(m·K), and decreases with carbon content: k=42 W/(m·K) for carbon steel with 1%C, k=32 W/(m·K) for carbon steel with 1.5%C, etc. Conductivity decreases with temperature in carbon steels (e.g. from 50 W/(m·K) at 300 K to 30 W/(m·K) at 1000 K). Conductivity increases with temperature in stainless steels (e.g. from 18 W/(m·K) at 300 K to 25 W/(m·K) at 1000 K). Going down, k=9 W/(m·K) at 100 K, k=0.7 W/(m·K) at 10 K). Conductivity decreases with alloying from k=26 W/(m·K) to k=15 W/(m·K).
Order of magnitude for liquids	1 (inorganic) 0.1 (organic)	Liquids are, in general poor conductors (see Liquid data), with k decreasing with temperature, in general. Liquid metals are good conductors: mercury has k=9 W/(m·K), and molten sodium has k=60 W/(m·K). The Na-K eutectic alloy (a room-temperature liquid with 22%wt Na) has k=23 W/(m·K) at 100 ºC.

		Thermal oils are not good on thermal conductivity (k=0.1 W/(m·K)) but on thermal stability (can work up to 600 K).
Water	0.6	Thermal conductivity of saturated water grows from k=0.57 W/(m·K) at 273 K to k=0.69 W/(m·K) at about 400 K, and then decreases to k=0.64 W/(m·K) at 500 K, k=0.50 W/(m·K) at 600 K, and k=0.24 W/(m·K) towards the critical point at 647 K. Water properties may be used as a first approximation for many natural aqueous solutions (milk, wine, beer, vinegar, seawater, urine, fruit juices, etc.)
Order of magnitude for gases	10^{-2}	Gases are very poor thermal conductors (see Gas data); hydrogen, with k=0.17 W/(m·K), and helium, with k=0.14 W/(m·K), are, by far the best conductors. According to the simplest generalised transport theory in gases, thermal diffusivity, mass diffusivity and kinematic viscosity of gases have the same values ($a=D_i=\nu\approx10^{-5}$ m^2/s). With this theory, gas thermal conductivity increases with the square root of temperature, and do not change with pressure; in reality, a linear dependence with temperature better fits experimental results.
Air	0.024	Thermal conductivity of foamed materials cannot be below that of air, unless the bubbles are air-tight and the foaming agent used CO_2 (k=0.015 W/(m·K)), R134a (CF_3CH_2F, k=0.014 W/(m·K)), or so.

Measuring k is based on measuring heat flux against temperature gradient in steady-state set-ups, or on measuring thermal diffusivity in transient processes. Measuring k in liquids is very difficult because test sizes are restricted to avoid heat convection, and it is even harder to measure k in gases because, not only convection must be avoided, but thermal radiation too. Many times, conductivity-values at different temperatures are needed, but notice that measuring k always implies a temperature difference to generate the heat flow, which cannot be too small without compromising uncertainty in the measure, thus all k-values are more or less averaged values..

A summary of the different approaches followed to measure thermal conductivities is presented below, quoting the formulation to be used, which is covered in detail separately under Heat Conduction modelling.

Steady state methods to measure thermal conductivity of materials (most accurate, but slow and expensive):

- Planar geometry. A slab (or a long isolated bar for good conductors) is kept between a guarded heater at one side of the sample (e.g. an electrical heater mat insulated on the other side, or in a symmetric configuration), and a guarded cooler on the other side of the sample (e.g. cooling water, or a thermoelectric refrigerator). If the heat flow is unidirectional, the conductivity is given by $k=\left(\dot{Q}/A\right)/\left(\Delta T/L\right)$, but, as it is very difficult to measure $\dot{Q}$ without losses, a reference-k material is used to measure k by comparison. Most accurate values for poor conductors are obtained in a

multilayer set-up like H-T-R1-T-S-T-R2-T-C, where H stands for the heater, T stands for an aluminium disc with embedded thermocouples, R1 and R2 stand for two reference-material discs (preferably with a similar conductivity as the sample), S stands for the sample, and C for the cooler. Very thin materials like films and foils are tested by stacking several sample layers together.

- Cylindrical geometry. A cylindrical sample with a central heater and a cooler on the outside is used. The conductivity is given by $k = \left(\dot{Q}/L\right)/\left(2\pi\Delta T/\ln\left(R_{\text{ext}}/R_{\text{int}}\right)\right)$, but, as above, a reference-k material can be used to measure k by comparison.

Unsteady methods to measure thermal conductivity of materials (not so accurate, but quicker (half a minute instead of half an hour), cheaper, and yield thermal diffusivity directly):

- Planar geometry. The most used variant of this method is the quasi-steady state method (Fitch-1935). A disc-shape sample (a thin slab may do) is sandwiched between a planar wall of a metal container with a well-stirred bath at a fixed temperature, T_{w}, and a small copper disc of mass m_{Cu}, and area A_{Cu}, well-insulated on all sides except that in contact with the sample. Assuming linear quasi-steady heat transfer through the sample, its conductivity is obtained by the copper energy-balance, $k = -\left(m_{\text{Cu}}c_{\text{Cu}}L/\left(A_{\text{Cu}}t\right)\right)/\left(\ln\left(\left(T_{\text{Cu}}(t)-T_{\text{w}}\right)/\left(T_{\text{Cu,initial}}-T_{\text{w}}\right)\right)\right)$; accuracy can be enhanced by optimisation of sample and copper thicknesses. Another planar method is based on the time lag between two thermocouples on each side of a thin sample, when a short light-pulse is shined on one side; this is known as flash method, and, although already introduced in the 1950s, still lacks accuracy; the thermal diffusivity of the sample is usually estimated as a=0.14$L^2/t_{1/2}$, where L is sample thickness, and $t_{1/2}$ is the time it takes for the rear thermocouple to reach half of the maximal temperature increase (temperature falls afterwards, due to heat losses, axially and laterally).
- Cylindrical geometry (also known as line heat source method). The method is based on the temperature rise at radius R within the sample, after a centred line heater of given power is switched on. The most used variant of this method is the singular case with R=0, i.e. when temperature rise is measured just at the axis, where the heater is also located; a line-heat-source probe holds both a heater and a thermometer, either inside a narrow tube, or on the outside of a tube or a rod (internal placement makes the probe more robust, but external placement yields more accurate data). The conductivity of the sample is obtained by fitting the straight portion of the $T(t)$ versus ln$t$ plot, and using $k = \left(\dot{Q}/(4\pi)\right)/\left(\mathrm{d}T/\mathrm{d}(\ln t)\right)$ (the curved initial and final portions should be discarded). The sample diameter D must be large enough, $D^2/(at)$>10, and probe diameter d narrow enough, say d/D<10, and even so, uncertainties are typically around 5%.
- Spherical geometry (also known as thermistor method). The method is based on measuring the temperature rise of a thermistor of radius R, encapsulated in a nearly-spherical bead, embedded within the sample, and used as a point source of constant power, $\dot{Q}$. The thermal diffusivity of the sample material is obtained from $\dot{Q} = \left(4\pi Rk + 4R^2\sqrt{\pi k\rho c/t}\right)\left(T_R(t)-T_\infty\right)$, i.e. by linear fitting of the thermistor temperature versus the inverse of the square root of time. This method required calibration with a medium of known thermal conductivity to find the effective bead radius, R, and even so, uncertainties are typically around 20%, mainly due to heat losses through the connectors..

Heat equation

The heat-transfer equation is the energy balance for heat conduction through an infinitesimal non-moving volume. To deduce it, we start from the energy balance (2) applied to a system of finite volume:

$$\left.\frac{dH}{dt}\right|_p = \dot{Q} \quad \rightarrow \quad \int_V \rho c \frac{\partial T}{\partial t} dV = -\int_A \vec{\dot{q}} \cdot \vec{n} dA + \int_V \phi dV \tag{4}$$

where ϕ is some energy release rate per unit volume (e.g. by nuclear or chemical reactions), sometimes written as $\dot{q}_{gen}$. Equation (4) can be read as "the time-increment of enthalpy within de volume is due to the heat input through the frontier plus the energy dissipation in the interior", the minus sign coming from the choice of $\vec{n}$ as the normal outwards vector. When the Gauss-Ostrogradski theorem of vector calculus is used to transform the area-integral to the volume-integral, and (4) is applied to an infinitesimal volume within the system, one gets:

$$\int_V \rho c \frac{\partial T}{\partial t} dV = -\int_A \vec{\dot{q}} \cdot \vec{n} dA + \int_V \phi dV = -\int_V \nabla \cdot \vec{\dot{q}}\, dV + \int_V \phi dV \quad \rightarrow$$
$$\xrightarrow{V \to 0} \quad \rho c \frac{\partial T}{\partial t} = -\nabla \cdot \vec{\dot{q}} + \phi \tag{5}$$

Finally, considering Fourier's law (3) and constant material properties (density ρ, thermal capacity c, conductivity k, and their combination, the thermal diffusivity:

$$a \equiv \frac{k}{\rho c} \tag{6}$$

one gets the so-called heat equation:

$$\rho c \frac{\partial T}{\partial t} = k \nabla^2 T + \phi, \quad \text{or} \quad \frac{\partial T}{\partial t} = a \nabla^2 T + \frac{\phi}{\rho c} \tag{7}$$

The heat equation (7) is the most well-known parabolic partial-differential equation (PDE) in theoretical physics, ϕ being a non-homogeneous term. The heat equation is also known as diffusion equation, and it has solutions that evolve exponentially with time to the steady state. At steady state, the heat equation becomes an elliptic PDE named Poison's equation, which, without the non-homogeneous term, becomes Laplace equation, $\tilde{N}^2 T = 0$. Besides parabolic and elliptic PDE, the third type is the hyperbolic PDE $\partial^2 \psi / \partial t^2 - c^2 \partial^2 \psi / \partial x^2 = 0$, typical of wave-like phenomena.

A more general heat equation takes account also of the effect of relative motion between the material system and the coordinate system with a velocity $\vec{v}$, which, in a fix reference frame (Eulerian reference frame) takes the form:

$$\frac{\partial T}{\partial t} = a\nabla^2 T + \frac{\phi}{\rho c} - \nabla \cdot \left(T\vec{v}\right) \tag{8}$$

although we will only consider such motions when analysing moving heat sources in a stationary solid (to change to a reference frame moving with the source), of application to machining, grinding, cutting, sliding, welding, heat treatment, and so many materials processing. The most general heat equation (e.g. to be used in computational fluid dynamics (CFD), must include in the energy release term ϕ, viscous dissipation and the dilatation work due to the time-variation of pressure along a fluid line, if any, although both are usually negligible energy contributions. To solve the heat equation, besides the parameters explicitly appearing in it (a, ϕ, ρ, c...), appropriate bounding conditions for the variables are required, i.e. initial conditions for time and boundary conditions for the space variables.

Modelling space, time and equations

Space-time modelling may refer to the consideration of continuous or discrete processes in space and time, but space-time modelling in heat transfer usually refers to the consideration of processes as steady or non-steady, zero-dimensional, one-dimensional, two-dimensional or three-dimensional geometry, planar, cylindrical or spherical, etc. So important this modellization is, that heat transfer books, and in particular the heat conduction part, is usually divided in different chapters for the different space-time models: steady one-dimensional conduction, unsteady one-dimensional conduction, steady two-dimensional conduction, etc.

As in many other engineering problems, the steady state solution is usually analysed first, in heat-transfer problems, leaving transient effects for a more advanced phase, but many undesirable events may occur during transients. Here in this context, thermal shock (e.g. breaking a glass by pouring hot water) and local overheating (e.g. charring shoes and cloth before getting warm), can be mentioned.

Every step in problem-solving may have an associated mathematical modelling, from geometrical definition and materials properties, to results and conclusions. We want now to consider the main mathematical tools used to formulate and solve heat-transfer problems, traditionally divided into classical analytical methods (partial differential equations in a continuum, developed in the 19th century), and modern numerical methods (discrete set of algebraic equations applied to small elements of the system, developed in the late 20th century, like the finite element method FEM, the finite difference method FDM, or the boundary element method BEM. In both cases there are imposed bounding conditions (BC), which, in heat transfer, are the initial conditions (heat equation is 1st order in time) and boundary conditions (heat equation is 2nd order in space variables); the latter are usually classified as:

- First kind boundary conditions, when the temperature is known at the boundary.
- Second kind boundary conditions, when the temperature-gradient is known at the boundary. It often happens that, on the whole boundary of a closed system, conditions of a first kind apply to some parts, and of the second kind to the others.
- Third kind boundary conditions, when the temperature-gradient at the boundary is a known function of the temperature there.

- Fourth kind boundary conditions, when neither the temperature-gradient nor the temperature-gradient are known at the boundary, but are functions of other boundary conditions in the problem (i.e. the boundary at hand is an intermediate boundary in a much larger system).

In practice, all real problems are of the third and fourth kind, because it is really difficult to force a constant temperature at a boundary (thermostatic baths and blocks are often used for the purpose), and even more difficult to guarantee a constant heat flux.

By the way, continuous field theory (as used in Fluid Mechanics, Elasticity, or Electromagnetism) is a simplifying recourse to modelling the influence of many discrete microscopic particles on other many-particles systems (i.e. the goal of Thermodynamics). Classical field theory started with Newton's Law of Gravitation, followed with Euler's Law of fluid motion (later expanded to Navier-Stokes equation), and Fourier's heat equation, and peaked with Maxwell's electromagnetism equations. But Fourier was the first to solve a multi-dimensional field equation (a PDE), inventing Fourier's series and separation of variables to solve the heat equation (he did it to win a prize offered by the French Academy).

Case studies

To better illustrate heat transfer theory and applications, we are considering the following two simple practical examples to throw light on the different approaches to solve heat transfer problems:

- Cooling-down of a hot sphere in a water bath (sphere-cooling, for short), where a glass sphere with D=1 cm in diameter, is taken out of a bath at T_1=100 ºC (e.g. boiling water) and submerged in a bath of ambient water at T_∞=15 ºC with an estimated convective coefficient of h=500 W/(m^2·K). We take for glass k=1 W/(m·K), ρ=2500 kg/m^3 and c=800 J/(kg·K). This is a one-dimensional, spherical-symmetry, unsteady problem of practical relevance in materials processing.
- Heating-up of a rod in ambient air by an energy source at one end (rod-heated-at-one-end, for short), where an aluminium rod of length L=0.1 m and diameter D=0.01 m, is being heated at one end with $\dot{Q}_0$=10 W (from an inserted electrical heater), while being exposed to a ambient air at T_∞=15 ºC with an estimated convective coefficient h=20 W/(m^2·K). We take for aluminium k=200 W/(m·K), ρ=2700 kg/m^3 and c=900 J/(kg·K), though k may vary from 120 W/(m·K) in the typical aerospace alloy (Al-7075) to 220 W/(m·K), in pure aluminium (Al-1100). This is a quasi-one-dimensional unsteady problem, which has a non-trivial steady state temperature profile. The name rod usually refers to centimetric-size elements; for much smaller rods, the word spine is more common, and the word beam for much larger elements.

These case studies may seem too stereotyped, but they are relevant to heat transfer practice, and they allow for comparison of practical numerical approaches with simple analytical exact solutions.

Nomenclature refresh

A science is a set of concepts and their relations. Good notation makes concepts more clear, and helps in the developments. Unfortunately, heat transfer notation is not universally followed.

	ISO 31-4 & 31-6	here
Heat flux [a)], dQ/dt	Φ	$\dot{Q}$
Heat flux density[b)], $dQ/(dt dA)$	ϕ or q	$\dot{q}$
Thermal conductivity, $dQ/(dt dA \Delta T)$	λ or k	k
Thermal conductance (or heat conductance), $dQ/(dt \Delta T)$, $G=1/R$	G	G
Thermal conductance coefficient (or heat transfer coeff.), $dQ/(dt dA \Delta T)$	K	K (old U)
Thermal resistance (or heat resistance), $R=1/G$	R	R
Thermal insulation coefficient (or heat insulatio coeff.), $M=1/K$	M	M
Thermal diffusivity, $k/(\rho c_p)$	a or α	a
Radiant energy, [J]	Q or W	(not used)
Radiant energy density, [J/m^3]	w	(not used)
Radiant energy flux, [W]	Φ or P	Φ
Incident radiations:		
- Irradiance (or incident (radiant energy) flux, direct plus diffuse), [W/m^2]	E	E (old i)
- Absorptance (at an opaque surface, or through transmitting media)	α	α
Emerging radiations:		
- Exitance (or emerging (radiant energy) flux), [W/m^2]	M	M
- Radiant intensity (or emerging radiant energy along a direction), [W/sr]	I	(not used)
- Radiance (or emerging radiant energy flux along a direction), [W/m^2·sr]	L	(not used)
- Emittance, [W/m^2], is not defined; only exitance. Planck law[c)] is $M(\lambda)$ for a blackbody. (exitance=emission+reflexion+transmission)	(only defined for a blackbody)	M
- Emisivity = real emittance / blackbody emittance	ε	ε

a) The desire to use a common symbol for all kind of fluxes is praiseworthy, but we still keep here to the traditional symbol $\dot{Q}$ for the heat flux, usually simplified to Q in most heat-transfer texts when there is no possible confusion between heat and heat flux.

b) In Heat Transfer, q and Q (instead of the here-used $\dot{q}$ and $\dot{Q}$) may be used for heat flux density and heat flux, respectively, to simplify notation, but $\dot{q}$ and $\dot{Q}$ are preferable when a Heat Transfer course is combined with Thermodynamics, to avoid symbol overriding.

c) Planck law: $M_\lambda = \dfrac{c_1}{\lambda^5\left[\exp\left(\dfrac{c_2}{\lambda T}\right)-1\right]}$, gives the spectral exitance of a blackbody.

OBJECTIVES OF HEAT TRANSFER (WHAT FOR)

Heat transfer theory may be used to compute heating/cooling times in heat transfer problems, or to compute temperature fields and heat fluxes, or to compute required dimensions or properties for heat insulation or conduction. In some special cases, the goal is to find the value of a parameter in a thermal problem that produces branching solutions, as in the onset of Bénard-Marangoni convection when heating a thin liquid layer from below (bifurcation analysis).

Heat-transfer problems may arise in typical thermal applications, like heating and cooling; e.g., the defrosting problem in refrigeration and air conditioning, due to thermal insulation of the ice layer (frosty ice

conductivity can be as low as that of wood), admits several solutions, all of them controlled by heat transfer. Any temperature-measure involves some heat transfer problem. Besides, many other heat-transfer problems come from non-thermal pursuits; for instance:

- Cooling electronic equipment. Microprocessor computing power is limited by the difficulty to evacuate the energy dissipation (a Pentium 4 CPU at 2 GHz in 0.18 μm technology must dissipate 76 W in an environment at 40 ºC without surpassing 70 ºC). Most electronics failures are due to overheating by improper ventilation or fan malfunction. Bipolar junctions in silicon wafers fail to keep the energy gap between valence and conduction electrons above some 400 K, but at any working temperature there is some dopant diffusion and bond-material creep, causing some random failures, with an event-rate doubling every 10 ºC increase; depending on the reliability demanded, bipolar junctions are usually limited to work at 90..100 ºC. Electrical powers up to a few watts can usually be dissipated by natural convection to ambient air, and up to few hundred watts by forcing air with fans (cheap, but noisy and wasteful), and liquid cooling is usually needed beyond 1 kW systems.
- Cooling rubbing parts. In a mechanical transmission, the oil loses its lubricating capacity if overheated; in a hydraulic coupling or converter, the fluid leaks under the pressure created. In an electric motor, overheating causes deterioration of the insulation. In an overheated internal-combustion engine, the pistons may seize in the cylinders.
- Lamp design. The size of an incandescent lamp is governed by heat transfer (the filament needs a bulb to keep away from oxygen, but the bulb-size is large in glass bulbs to avoid glass softening by high temperatures, and small in halogen lamps to be hot enough to maintain the halogen cycle inside (to avoid deposition at a cold bulb), in spite of the little size for the electrical resistance R that must be fed at low voltage for the same power $P=VI=V^2/R$).
- Materials processing like casting, welding, hot shaping, crystal growth, etc. Materials machining is limited by the difficulty to evacuate the energy dissipation. And not only engineering materials: food processing and cooking, dish washing, cloth washing, drying and ironing, and many other house-hold tasks, are dominated by heat transfer.
- Energy conversion devices, like solar collectors, combustors, nuclear reactors, etc.
- Environmental sciences like meteorology, oceanography, pollutant dispersion, forest fires, urban planning, building, etc.

Relaxation time

The two usual limits for thermal interaction in Thermodynamics are the isothermal process and the adiabatic process, the former corresponding to the limit of very slow heat transfer due to an infinitesimal temperature difference along an infinite time, and the latter corresponding to a quick process with negligible heat transfer. In Heat Transfer, however, we are interested on finite-time process, and a basic question is to know the thermal inertia of the system, i.e. how long the heating or cooling process takes, usually with the intention to modify it, either to make the system more permeable to heat, more insulating, or more 'capacitive', to retard a periodic cooling/heating wave.

When the heat flow can be imposed, as when heating water with a submerged electrical resistor, the minimum time required is obtained from the energy balance, $\mathrm{d}H/\mathrm{d}t = \dot{Q} = mc\Delta T/\Delta t$; e.g. to heat 1 kg of water from 15 ºC to 95 ºC with a 1000 W heater, the minimum time is $\Delta t = mc\Delta T/\dot{Q}$=1·4200·(95-15)/1000=336 s (the actual value in practice depends on the way of heating, the geometry of the vessel, and the way temperature is measured; typically, 30% more time is required with highly efficient types of heaters like a microwave oven or induction heaters, and up to 100% more time with inefficient heaters as external electrical resistors).

For the case where the heat flux is not imposed but a temperature gradient is imposed, an order-of-magnitude analysis of the energy balance, $\mathrm{d}H/\mathrm{d}t = \dot{Q} \rightarrow mc\Delta T/\Delta t = KA\Delta T$, shows that the relaxation time is of the order $\Delta t = mc/(KA)$, and, depending on the dominant heat-transfer mode in K, two extreme cases can be considered: solids in well-stirred fluids (convection dominates, and evolution is driven by conduction), and highly conducting solids (convection-driven case).

Conduction driven case (convection dominates)

Problems where the thermal conductance from a solid system to a surrounding fluid, $K=h$, is much larger than the thermal conductance within the solid, $K \approx k/L$, i.e. where $Bi \equiv hL/k \rightarrow \infty$ (Bi is a non-dimensional parameter called Biot number). In this case, the boundary condition imposes a constant temperature on the body surface, and the time Δt it takes for heat to penetrate to the centre of the body, of characteristic length L (volume divided by surface), i.e. the relaxation time may be guessed from (2-7):

$$\Delta t \approx \frac{\Delta H}{\dot{Q}} \approx \frac{mc\Delta T}{KA\Delta T} \approx \frac{mc\Delta T}{kA\dfrac{\Delta T}{L}} \approx \frac{\rho L^3 c}{kL^2\dfrac{1}{L}} = \frac{\rho c L^2}{k} = \frac{L^2}{a} \tag{9}$$

where the ΔT from initial to final states of the system has been assumed to be of the same magnitude of the representative ΔT from the system to the surroundings (although the former is not uniform and the latter is not constant but decreases with time). Thus, the time it takes for the centre to reached a mid-temperature representative of the forcing is L^2/a, i.e. increases with the square of the size, decreases with thermal diffusivity, and is independent of temperature.

- Exercise 1. Make an estimation of the time for the 1 cm glass-ball to cool down, in our sphere-cooling problem stated above.
  Solution. In this case $Bi=hL/k$=500·(0.01/6)/0.6=1.5>1, large enough, and the time it takes for the centre to reach a representative temperature of the heating process (e.g. a mid-temperature between the initial and the final, say 60 ºC), is $\Delta t=\rho c L^2/k$=2500·800·$(0.01/6)^2$/1=6 s, where the characteristic length of a spherical object, $L=V/A=(\pi D^3/6)/(\pi D^2)=D/6$, has been used.
  Could this model be applied to the cooling down of a hot potato in air?
  Notice that this convection-dominated model is not applicable to our rod-heated-at-one-end problem, since, for it $Bi=hL/k$=20·(0.01/6)/200=10^{-4}<<1 (neither to the hot potato in air. $Bi=hL/k$=10·(0.06/6)/0.6=0.17<1).
- Exercise 2. Make an estimation of the time it takes to boil an egg.

Solution. Assuming an egg at 5 ºC (from the fridge) is suddenly put into boiling water at 100 ºC (a bad practice since the jiggling might break the shell against the walls; it is better to keep it below boiling), and neglecting chemical energy changes, the time it takes for the centre to reach mid temperatures (say 50 ºC), is $\Delta t=\rho c L^2/k$=1000·4000·(0.04/6)^2/0.6=300 s (i,e, about 5 minutes), where the characteristic length of a nearly spherical object, $L=D/6$ has been used, with an egg diameter of D=4 cm, and thermal properties of water (most food, as the human body itself, have a water content of some 70%). This is a convection-dominated problem since the convective coefficient is of order h=1000 W/(m^2·K) and thus $Bi=hD/k$=1000·(0.06/6)/0.6=17>>1.
Three levels of egg boiling may be distinguished: suck-egg (also named egg from the shell, or *oeuf à la coque*), boiled for about two minutes, what leaves it semi-liquid throughout; soft-boiled egg, boiled for 3 to 5 minutes, what leaves a barely solid outer white, a milky inner white and a warm yolk, to eat with a spoon from the shell; hard-cooked egg, boiled for 10 to 15 minutes, what leaves a solid to be peeled and consumed apart. When heating an egg, the globular-folded aminoacids in albumin (a sol dispersion) start to unfold and stick, becoming less fluid and less transparent, forming a gel (a porous network of interconnected solid fibres spanning the all the volume of a liquid medium). Egg white starts to coagulate at about 60 ºC and ends at 65 ºC, when proteins denaturize; the yolk has different proteins and more fat, and coagulates from 65 ºC to 70 ºC.

It is important to be aware of the huge increase in heat transfer rate that even a small fluid convection may bring. If air inside a room of size L=3 m were to be heated with a radiator just by heat conduction (without air motion), a time $\Delta t=L^2/a=3^2/10^{-5}=10^6$ s (i,e, about 10 days!) would be required, whereas in the real case, the heat-up time may be estimated as $\Delta t=L/v$, where v is an average speed of the natural convection due to the draught caused by the heated air close to the radiator; if we estimate the air velocity in its vicinity from the draught pressure balance, $\Delta\rho g\Delta z=(1/2)\rho v^2_{\max}$, with $\Delta\rho=\rho\alpha\Delta T=\rho\Delta T/T$, we get $v_{\max}=(2g\Delta z\Delta T/T)^{1/2}$=0.8 m/s for a typical radiator height of Δz=1 m, heating some 10 K the surrounding air at 300 K; if we take as room-average speed $v=v_{\max}\delta/L$=0.8·0.1/3=0.03 m/s (having assumed a draught thickness of δ=0.1 m), the heating time (by natural convection) is now $\Delta t=L/v=3/0.03=10^2$ s, not a bad guess (10^3 s is a more realistic order of magnitude, but nothing comparable to the 10^6 s we got in the diffusion case).

Convection driven case (conduction dominates)

Problems where the thermal transmittance within the system, $K=k/L$, is much larger than the thermal transmittance to the surroundings, $K=h$, (i.e. where $Bi\equiv hL/k\to 0$). In this case, the temperature within the body can be assumed spatially uniform and the relaxation time may be guessed from (2-7):

$$\Delta t \approx \frac{\Delta H}{\dot{Q}} \approx \frac{mc\Delta T}{KA\Delta T} \approx \frac{mc}{hA} \approx \frac{\rho L^3 c}{hL^2} = \frac{\rho c L}{h} \qquad (10)$$

where, as before, the ΔT from initial to final states of the system has been assumed to be of the same magnitude of the representative ΔT from the system to the surroundings. The difference now is that the convective coefficient with the environment is not a material property but depends a lot on the motion outside, and that now the relaxation time is directly proportional to the size L and not its square function.

- Exercise 3. Make an estimation of the time it takes for the rod to reach steady state, in our rod-heated-at-one-end problem.
  Solution. For a rod with lateral convection, $Bi=hL/k=20\cdot(0.01/4)/200=2.5\cdot10^{-4}<<1$, where the and thence the time it takes to heat up (for the centre to reach a mid temperature between the initial and the steady state), is $\Delta t=\rho cD/h=2700\cdot900\cdot(0.01)/20=1200$ s (i,e, some 20 minutes), where the characteristic length of a cylindrical object, $L=V/A=(\pi D^2/4)/(\pi D)=D/4$ has been used, instead of the axial length, which is of little relevance in the transient heat up.
- Exercise 4. Make an estimation of the time it takes for a hot potato to cool down.
  Solution. Assuming a baked potato is brought out of an oven at 100 ºC (mind that even after 1 hour in an oven at 250 ºC, food can only reach some 100 ºC while it is moist), and left in air at 20 ºC, the time it takes for the centre to reach a representative temperature of the cooling process (e.g. a mid-temperature between the initial and the final, say 60 ºC), is $\Delta t=\rho cL/h=1000\cdot4000\cdot(0.06/6)/10=4000$ s (i,e, more than 1 hour!), where the characteristic length of a nearly spherical object, $L=D/6$ has been used, with a potato diameter of D=6 cm, the thermal properties of water (most food, as the human body itself, have a water content of some 70%), and a typical value of the convective coefficient in calm air of h=10 W/(m^2·K). We have neglected chemical energy changes, but the result obtained is longer than experience tells (say, over half an hour), mainly due to radiation losses (wrapping a hot potato in aluminium paper, a very poor emitter, keeps it hot for nearly one hour). Notice that potatoes should never be forced-cooling by water immersion (unlike most other vegetables) because they would get moist. Moreover, potato skin should be pierced before baking to let vapour escaping. Potatoes loss weight on baking (oven and microwave): from 10% the large ones to 25% the small ones. Can you now predict how long will it take for a frozen turkey of 5 kg to thaw when taken out of the freezer? (Beware of the phase change.)

Heat flux

Thermodynamics is concerned with heat accounting $Q=\Delta E-W$, but Heat Transfer is focussed on rates, so that one question is to know the relaxation time for a given amount of heat (see before), and another question is the flow-rate function considered here (yet, a third one would be to find the materials and geometry that satisfy a prescribed flow-rate or relaxation time; see below).

Typical heat-flux problems mostly consist on finding the heat flux corresponding to prescribed temperatures at the boundaries of a given geometry and known material. In many circumstances, the simple steady state, one-dimensional planar geometry, may be a good approximation, and in other cases the uniform-temperature approximation is acceptable; for more complicated problems, the heat-flux problem must be found concurrently with the temperature field (next point). A couple of simple applications follow.

- Exercise 5. Find the heat flux through a composite wall, with application to the bottom of a freezer boat with -30 ºC cargo temperature, 1 mm thick stainless steel lining sheet, 10 cm thick polyurethane insulation, 1 cm thick air layer, and 1 cm thick steel of hull in contact with water at 10 ºC.

Solution. Assuming steady state in one-dimensional planar geometry, the heat flux must be the same through every layer, so that for the combined sandwich:

$$\dot{q} = K\Delta T = k_{12}\frac{T_2 - T_1}{L_{12}} = k_{23}\frac{T_3 - T_2}{L_{23}} = \ldots = \frac{T_n - T_1}{\sum \frac{L_i}{k_i}} \quad \Rightarrow \quad K = \frac{1}{\sum \frac{L_i}{k_i}} \tag{11}$$

where the interfaces are named 1,2,…,n; i.e., the overall thermal transmittance is given by (11), best recalled in terms of their inverse, the thermal resistance coefficient R=1/K, what establishes that the overall resistance is the sum of the partial resistances $R=\sum R_i=\sum L_i/k_i$ (electrical analogy of a circuit with resistances in series). For the numerical example here:

$$\dot{q} = \frac{T_n - T_1}{\sum \frac{L_i}{k_i}} = \frac{(10+273)-(-30+273)}{\frac{0.01}{52}+\frac{0.01}{0.024}+\frac{0.1}{0.027}+\frac{0.001}{20}} = \frac{40}{0.0002+0.42+3.7+0} = 9.7\ \text{W/m}^2 \tag{12}$$

where thermal conductivity values for steel, air, polyurethane, and stainless steel of (52, 0.024, 0.027, and 20) W/(m·K), respectively, have been assumed.

- Exercise 6. Find the heat transfer rate during the cooling down of the glass ball in our sphere-cooling problem stated above.
  Solution. If uniform temperature inside the ball could be assumed, the energy balance would yield an ordinary differential equation, easily integrated:

$$\dot{Q} = \frac{dH}{dt} = KA\Delta T \longrightarrow mc\frac{dT}{dt} = -hA\left(T - T_\infty\right) \quad \Rightarrow \quad \frac{T - T_\infty}{T_0 - T_\infty} = \exp\left(-\frac{hA}{mc}t\right) \tag{13}$$

showing an exponential temperature decrease and a corresponding exponentially decreasing heat flow-rate, that for the numerical values of the glass ball cooling in water gives

$$\frac{T - T_\infty}{T_0 - T_\infty} = \exp\left(-\frac{hA}{mc}t\right) = \exp\left(-\frac{6h}{\rho Dc}t\right) = \exp\left(-\frac{6\cdot 500}{2500\cdot 0.01\cdot 800}t\right) = \exp\left(-\frac{t}{7\ \text{s}}\right) \tag{14}$$

and

$$\dot{Q} = -hA\left(T - T_\infty\right) = -h\pi D^2\left(T_0 - T_\infty\right)\exp\left(-\frac{hA}{mc}t\right) = -\left(13\ \text{W}\right)\exp\left(-\frac{t}{7\ \text{s}}\right) \tag{15}$$

i.e., the water bath starts getting 13 W from the ball, but at a decreasing rate insomuch that $\int \dot{Q}dt = \rho c V\Delta T = 2500\cdot 800\cdot(\pi 0.01^3/6)\cdot(100-15) = 89$ J (in effect, (13 W)·(7 s)≈90 J).

Exercise 7. Dew on window panes

Temperature field

Although in many thermal problems (and even in some heat transfer problems) temperature is assumed uniform in a system (as in the hot-potato problem just explained, and in many heat convection and radiation problems), temperature is never uniform in practice (equilibrium systems are just limit models; we live in a non-equilibrium world). The heat equation (4), with the appropriate initial and boundary conditions, can provide a spatial temperature distribution at every time.

Many times, particularly in thermal control problems, only the extreme temperatures are sought. Living beings have very short temperature ranges (human cannot support body temperatures outside T=(310±4) K, and typical electrical batteries deteriorate outside T=(300±30) K. Refractory materials are those material able to withstand high temperatures (say >1500 K), without failure by fusion or decomposition; they are usually classified according to chemical behaviour, as acid refractories (fireclay), neutral refractories (coal, graphite, refractory metals and metal carbides), and basic refractories (metal oxides).

Two examples of temperature field computations are presented below:

- Exercise 8. When temperature at a surface of a semi-infinite solid (of constant properties, initially at T_∞) is suddenly brought to T_0), the disturbance propagates to a depth x in a time t such that the temperature profile (one-dimensional because of the initial and boundary conditions), and heat flux density, are given respectively by:

$$\frac{\partial^2 T}{\partial x^2}-\frac{1}{a}\frac{\partial T}{\partial t}=0 \xrightarrow{\eta\equiv\frac{x}{2\sqrt{at}}} \frac{\partial^2 T}{\partial \eta^2}+2\eta\frac{\partial T}{\partial \eta}=0 \rightarrow T=c_1+c_2\mathrm{erf}(\eta)$$
$$\frac{T(x,t)-T_0}{T_\infty-T_0}=\mathrm{erf}\left(\frac{x}{2\sqrt{at}}\right) \quad \dot{q}(x,t)=-k\frac{(T_\infty-T_0)}{\sqrt{\pi a t}}\exp\left(\frac{-x^2}{4at}\right) \tag{16}$$

  a being the thermal diffusivity of the material. Notice the high idealisation of the statement: semi-infinite extent (in one dimension, infinite in the other two), and infinitely quick temperature jump, what makes the problem of little practical use, but notice the simplicity of the result too (practical heat-transfer problems usually end up with a myriad of numbers difficult to crunch, or with a multiplicity of partial graphics difficult to integrate).

- Exercise 9. Find the steady temperature profile in our rod-heated-at-one-end problem, with a prescribed temperature value at one end, instead of the constant heat source.
  Solution. We can consider this problem a one-dimensional heat conduction problem axially, with internal heat sinks to account for the actual lateral heat losses by convection (i.e. with $\phi A\mathrm{d}x=-hp\mathrm{d}x(T-T_\infty)$, p being the perimeter and A the cross-section area), and apply (7) to an infinitesimal slice:

$$\rho c A\mathrm{d}x\frac{\partial T}{\partial t}=kA\mathrm{d}x\nabla^2 T-hp\mathrm{d}x(T-T_\infty) \xrightarrow{t\to\infty} 0=\frac{\mathrm{d}^2 T}{\mathrm{d}x^2}-\frac{hp}{kA}(T-T_\infty) \tag{17}$$

and, imposing the boundary conditions:

$$\left.\begin{array}{l} 0=\dfrac{\mathrm{d}^2T}{\mathrm{d}x^2}-\dfrac{hp}{kA}\left(T-T_\infty\right) \\ T\big|_{x=0}=T_0 \\ 0=-kA\dfrac{\mathrm{d}T}{\mathrm{d}x}\Big|_{x=L} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \dfrac{T(x)-T_\infty}{T_0-T_\infty}=\dfrac{\cosh\left[m\left(L-x\right)\right]}{\cosh\left(mL\right)} \\ \dot{Q}\left(x\right)=\sqrt{hpkA}\left(T_\infty-T_0\right)\dfrac{\sinh\left[m\left(L-x\right)\right]}{\cosh\left(mL\right)} \end{array}\right. \tag{18}$$

where $m\equiv\sqrt{hp/(kA)}$, h being the convective coefficient, p the perimeter of the rod cross-section (πD if circular) , A the cross-section area ($\pi D^2/4$ if circular), and L the rod length.

Dimensioning for thermal design

The goal of most heat transfer modelling is to find the temperature field and heat fluxes in a material domain, given a set of constraints: general heat equation (e.g. set as a partial differential equation, PDE), boundary conditions (BC), initial conditions (IC), distribution of sources or sinks (SS), etc. In a few cases the goal is not in the direct problem (given the PDE+BC+IC+SS, find the temperature field), but on the inverse problem: given the T-field and some aspects of PDE+BC+IC+SS, find some missing parameters remaining (identification problem).

Perhaps the very simplified, yet very important, problem of one-dimensional steady heat transfer between two bodies, separated by a solid layer, can make more clear the several different goals in heat transfer: heat fluxes, T-fields, material characterisation, and dimensioning:

- $\dot{Q}=kA\left(T_1-T_2\right)/L$, i.e. find the heat flux for a given set-up and T-field.
- $T_1=T_2+\dot{Q}L/\left(kA\right)\right)$, i.e. find the temperature corresponding to a given heat flux and set-up. Notice that our thermal sense (part of the touch sense) works more along balancing the heat flux than measuring the contact temperature, what depends on thermal conductivity of the object; that is why Galileo masterly stated that we should ascribe the same temperature to different objects in a room, like wood, metal, or stone, contrary to our sense feeling.
- $k=\dot{Q}L/\left(A\Delta T\right)$, i.e. find an appropriate material that allows a prescribed heat flux with a given T-field in a given geometry.
- $L=kA\left(T_1-T_2\right)/\dot{Q}$, i.e. find the thickness of insulation to achieve a certain heat flux with a given T-field in a prescribed geometry.

Other typical example of thermal design follows.

- Exercise 10. Find the minimum conductivity for a pot handle of length L=0.2 m and A=1 cm^2 cross-section, to avoid hand-burning (assume T_{burn}=45 ºC) when holding the handle up to the middle while the end at the pot is at boiling-water temperature.
  Solution. We start assuming that the hand is not modifying the thermal problem; i.e., we want to find when we have T_{burn} at $L/2$. A first analysis shows that a key point in the thermal problem is missing:

what causes temperature to fall along the handle? The answer is, of course, heat losses to ambient air by convection, which should be modelled. Assuming ambient air at T_∞=20 ºC and a convective coefficient of h=10 W/(m^2·K), one may establish the desired relation from (18):

$$\frac{T_{\text{burn}}-T_\infty}{T_0-T_\infty}=\frac{\cosh(mL/2)}{\cosh(mL)} \quad \rightarrow \quad \frac{45-20}{100-20}=\frac{\cosh(mL/2)}{\cosh(mL)} \quad \rightarrow \quad \frac{hp}{kA}L^2=\text{constant} \quad (19)$$

where $m \equiv \sqrt{hp/(kA)}$ has been substituted, to reach the conclusion that the allowed conductivity increases with h, p and L (of course, when more convection or longer handle, more conductive handles can be allowed), and decreases with A (the larger the cross-section, the most insulating the handle material must be). Notice that, for a given area, larger perimeter handlers are best. Assuming a square solid handle, the above constant has a value of 6.12, m=12.4 m, and the maximum allowable handle conductivity is k=26 W/(m·K), i.e. a stainless-steel handle can be allowed (from Thermal data of solids, k=16..26 W/(m·K), depending on the type). In most cases, however, non-metal handles are implemented, a good reason being that the user tends to hold the handle much closer to the pot root, to decrease the force moment.

PROCEDURES (HOW IT IS DONE)

Thermal design

Design is an intricate multidisciplinary top-down activity (see Thermal Systems, for an overview). Thermal design, in heat-transfer problems, aims at providing a suitable configuration (materials, components, geometry, arrangement...), amongst different possibilities, trying to optimise the cost/benefit. For instance, a thermal designer may be asked to provide solutions to keep a computer CPU dissipating 70 W without becoming hotter than 70 ºC; amongst the different possibilities, the most common one nowadays is to leave some free-room nearby and blow air with a fan (with the associated noise and dissipation increase), but using a heat-pipe to efficiently-connect the internal chip with an external ample sink is already taking over (e.g. in laptops); high-power-dissipation devices may demand liquid cooling loops or even phase change loops (which might be expandable in some cases, similar to animal sweating).

Thermal design requires a broad knowledge of the subject (and related subjects), and is left to a later stage in training, except for simple 'design' problems where the configuration is already given and only a parameter of the configuration is to be optimised. The most common endeavour for beginners is to solve well-defined thermal problems, i.e. to perform some heat transfer analysis to find temperatures, heat fluxes, or relaxation times.

Thermal analysis

To solve a heat-transfer problem in practice, to find the temperature field and heat fluxes, like for any other engineering task, there are not magic recipes, but sound understanding of the subject matter. The practitioner should not compile a set of graphics, tables and formulas, much less the student. On the contrary, they should master the principles of heat transfer, and have an idea of the different tools available.

Several steps are usually taken to solve a heat-transfer problem:

1. Mathematical modelling of the physical problem. This is the most creative phase in solving a problem. Sometimes, physical analogies help to build the mathematical model. The electrical analogy consists on taking thermal resistances as resistors, thermal capacities as capacitors, heat sources as intensity sources, the thermal environment as electrical ground, and applying Kirchhoff's law to the network. It is important, however, to keep in mind that analogies are just analogies, not identities, and care is needed to avoid stretching them beyond their applicability.
2. Mathematical solution of the mathematical problem. Although it is just a mathematical burden, engineers must be aware of the available methods of solution, and their pros and cons, in order to direct the previous idealisation towards feasible, available, affordable, efficient and solvable problems. The two basic approaches are:
   - Analytical solutions, which gives a whole and concise parametric solution, but only in extremely idealised problems (only of academic interest or to check numerical simulations).
   - Numerical solutions, which gives particular solutions to any practical problem, but without an overview of the influence of the parameters (several particular cases must be solved to have an idea of the influences).
3. Analysis of the results (analytical or numerical) and physical interpretation. In some circumstances, particularly with new or complicated problems, some experimental tests, where the temperature field and heat fluxes are metered in an instrumented sample, are required to provide evidence of the goodness of the mathematical modelling.

In actual practice, heat-transfer problems are solved numerically by using a large commercial computer package, usually an integrated fluid-thermal-structural CFD-package, or at least with inputs and outputs compatible with main commercial packages for mechanical and structural analysis.

Mathematical modelling

The mathematical modelling is the idealisation of the physical problem until a well-defined set of (mathematical) constraints, representing the main features, is established. Mathematical modelling is required not only in analytical work but also in actual heat-transfer practice, where a large commercial computer package is used; the user has to identify and approximate the actual geometry of the system, has to select the most appropriate terms from the list of supplementary effects in the PDE, must approximate the boundary conditions according to specific package procedures, and, most important of all, the user has to give knowledgeable feed-back on possible weaknesses and improvements, since heat-transfer analysis, as any other engineering activity, is an iterative process that must be refined as needed; effort proportional to expected utility (a common error of beginners, both at school and at work, is to spend too much effort and time pursuing very precise numerical solutions to 'what if' preliminary problems that are discarded soon afterwards, or even before being finished!).

Mathematical modelling is the most creative part in the whole process of solving heat-transfer problems. Modelling usually implies approximating the geometry, materials properties, and the heat transfer equations.

Modelling the geometry

In thermal problems, the first task is to identify the system under study. On one side, the geometry is idealised, assuming perfect planar, cylindrical or spherical surfaces, or a set of points and a given interpolation function. Besides the edges or boundaries (which are usually fixed, as in Fig. 1, except in some special cases like the Stefan problem of moving phase-change), further information is needed to know if the region or domain of interest lies inside, outside, or in between boundaries. Additionally, several numerical methods of solving heat-transfer problems, make use of a subdivision of the domain in small sub-domains called elements, and procedures are needed to carry out an automatic meshing and the associated numbering. Location procedures are also needed to know to which element a given point belongs, which are the neighbour elements, and so on.

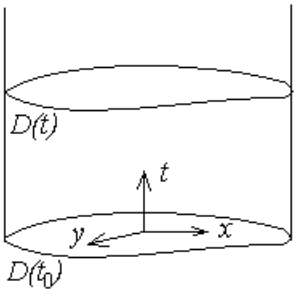

Fig. 1. The space-time domain is divided in the spatial domain or boundary, D (that may be one-, two- or three-dimensional, and is usually assumed independent of time in thermal problems, $D(t)=D(t_0)$), and the time domain (that is one-dimensional, with a clear start, $t=t_0$, and a clear bias, $t>t_0$).

The most complicated case occurs when boundary conditions are imposed on free-moving boundaries, i.e. surfaces with a priori unknown locations which separate geometric regions with different characteristics, as in heat-transfer problems with phase change; e.g. freezing of liquids or moist solids, casting, or polymerisation. This type of moving-boundary-value problems is known as Stefan problem, because Jozef Stefan was the first, in 1890, to analyse and solve it, when studying the rate of ice formation on freezing water, although a similar problem was first stated in 1831 in a paper by Lamé and Clapeyron. Phase-change materials are very efficient thermal-energy stores, either to accommodate heat input to heat output, or even to get rid of large amounts of thermal energy by ablation. In the normal case of phase-change accumulators, only the solid/liquid phase-change is considered, and with some buffering space to avoid large pressure build-up; this void fraction, plus the usual metal mesh used to increase thermal conductance, makes thermal modelling complicated.

- Exercise 11. Find the time for a liquefied-nitrogen-gas pool, 4 mm thick, to vaporise when suddenly spread over ground.

  Solution. The problem of spreading and vaporisation of cryogenic liquids, when there is a spillage over ground or water, is similar to the problem of water pouring over a very hot plate. Initially, the temperature jump is so large that there is a violent vaporisation at the contact surface, with formation of a thin (say tenths of a millimetre) vapour layer in between that isolates the liquid from the solid. Even with this vapour resistance, the solid starts to cool down, until the temperature jump is not enough to generate the vapour layer, which collapses and brings the liquid directly in contact with the

solid, increasing very much the solid cooling-rate, and changing the vaporisation from film boiling to nucleate boiling (see Heat transfer with phase change). This phenomenon was first described by J.G. Leidenfrost, in 1756, and is named after him. If we here disregard the initial vapour layer, and consider a uniform liquid layer of initial thickness L, vaporising at a rate controlled by the heat flux being supplied from the ground, which is modelled as a semi-infinite solid with a fixed temperature-jump at the surface (see Similarity solutions in Heat conduction), the energy balance gives:

$$\dot{q}_0 = \frac{\dot{m}_{vap} h_{LV}}{A} = -\rho \frac{\mathrm{d}L}{\mathrm{d}t} h_{LV} \tag{20}$$

ρ and h_{LV} being the density and vaporisation enthalpy of the liquid, whereas the heat flux is (Case 1 from Table 6 in Heat conduction):

$$\dot{q}_0 = k \frac{\Delta T}{\sqrt{\pi a t}} \tag{21}$$

k and a being the thermal conductivity and diffusivity of the solid, and ΔT the constant temperature jump form the liquid to the solid far away.

The solution is then:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{k}{\rho h_{LV}} \frac{\Delta T}{\sqrt{\pi a t}} \longrightarrow L = L_0 - \frac{2k\Delta T}{\rho h_{LV}} \sqrt{\frac{t}{\pi a}} \quad \Rightarrow \quad t_0 = \pi a \left(\frac{L_0 \rho h_{LV}}{2k\Delta T} \right)^2 \tag{22}$$

L_0 being the initial layer thickness and t_0 being the time for the whole layer to vaporise (when $L(t)=0$). Substituting numerical values for liquid methane (as an approximation to LNG mixture, from Liquid property data), $a=k/(\rho c)=0.18/(423\cdot 3480)=0.12\cdot 10^{-6}$ m^2/s, ρ=423 kg/m^3, h_{LV}=510 kJ/kg, k=0.18 W/(m·K), c=3480 J/(kg·K), with L_0=4 mm and $\Delta T=T_0-T_b$=288-112=176 K, we finally have t_0=70 s, i.e. about one minute.

One should keep in mind that real applications usually have complex geometry, with different materials (e.g. thermal problems in electronic boards), and the fact that a good modelling should only retain key thermal elements with approximated shapes, as major heat dissipaters with box or cylindrical shapes, and most sensitive items (e.g. oscillators, batteries). Most of the times, the geometry, material and boundary conditions are such that real 3D problems can be modelled as 2D or even 1D, with immense effort-saving.

Modelling materials properties

Once the system is defined, its materials properties must be idealised, because density, thermal conductivity, thermal capacity, and so on, depend on the base materials, their impurity contents, actual temperatures, etc. (see Table 1 above.) Most of the times, materials properties are modelled as uniform in space and constant in time, for each material, but, whether this model is appropriate, or even the right selection of the constant-property values, requires insight.

Unless experimentally measured, thermal conductivities from generic materials may have uncertainties of some 10%. Most metals in practice are really alloys, and thermal conductivities of alloys are usually much lower than those of the components, as shown in Table 2; it is good to keep in mind that conductivities for pure iron, mild steel, and stainless steel, are (80, 50, 15) W/(m·K), respectively. Besides, many common materials (like graphite, wood, holed bricks, reinforced concrete), are highly anisotropic, with directional heat conductivities, particularly all modern composite materials. And measuring k is not simple at all: in fluids, avoiding convection is difficult; in metals, minimising thermal-contact resistance is difficult; in insulators, minimising heat losses relative to the small heat flows implied is difficult; the most accurate procedures to find k are based on measuring thermal diffusivity $a=k/(\rho c)$ in transient experiments.

Table 2. Thermal conductivities of some typical alloys and its elements.

Alloy	k [W/(m·K)] of alloy	k [W/(m·K)] of element	k [W/(m·K)] of element
Alu-bronze C-95400	59	393 (Cu)	220 (Al)
(10% Al, >83% Cu, 4% Fe, 2% Ni)			
Mild steel G-10400	51 (at 15 ºC)	80 (Fe)	2000 (C, diamond)
(99% Fe, 0.4% C)	25 (at 800 ºC)		2000 (C, graphite, parallel)
			6 (C, graphite, perpend.)
			2 (C, graphite amorphous)
Stainless steel S-30400	16 (at 15 ºC)	80 (Fe)	66 (Cr)
(18.20% Cr, 8..10% Ni)	21 (at 500 ºC)		90 (Ni)

Unless experimentally measured, convective coefficients computed from generic correlations may have uncertainties of some 10%, whereas those taken from 'typical value' tabulations are just coarse orders of magnitude, e.g. when it is said that typical h-values for natural convection in air are 5..20 W/(m^2·K) and one assumes h=10 W/(m^2·K).

Unless experimentally measured on the spot, absorptance coefficients and emissivities of a given surface can have great uncertainties, which in the case of metallic surfaces may be double or half, due to minute changes in surface finishing and weathering.

Modelling the heat equations

The equations defining a heat-transfer problem, in systems where thermal conduction is the only heat-transfer mechanism in the interior, are the heat equation (5), and its bounding conditions (initial and boundary conditions). In systems with internal convection, the above equations must be solved concurrently with the fluid mechanics equations of Navier-Stokes. In systems with internal radiation, very complicated integral-differential equations appear when one considers spectral absorptances and multidirectional dispersions. Here we restrict the rest of the analysis to conductive systems, with convective and/or radiative effects entering only as boundary conditions.

There are a number of commercial packages for numerical solutions of PDE (like NASTRAN), applicable in principle to thermal, structural, fluid and electrical problems. However, in practice, the thermal problem may be highly non-linear (particularly if radiation is important) and it may be inconvenient to use the same discretization or even the same problem for thermal and structural analysis (in many cases the number of nodes and elements is 1 to 2 orders of magnitude larger for structural than for thermal analysis) To use these commercial packages, the user first makes use of a pre-processor (included in the package or dedicated ones like MSC/Patran or SCRC/Ideas) to draws the geometry or to import it as a CAD-file, to defines the materials (from a pre-loaded list or entering its properties), and to indicates a mesh type and size, what, together with and the specification of the particular boundary conditions (what is usually the hardest task), completes the input to the solver. After some time (always longer than expected) the solver produces a huge amount of information (output from the solver) that must (always) first be checked out for validity, before any further analysis. The user needs a post-processor (included in the package or a dedicated one like MSC/Patran or SCRC/Ideas) to interpret the results.

Perhaps the key point to remember when actually doing the mathematical modelling of thermal problems is that it is nonsense to start demanding great accuracy in the solution when there is not such accuracy in the input parameters and constraints. Without specific experimental tests, there are big uncertainties even in materials properties, like thermal conductivity of metal alloys, entrance and blocking effects in convection, and particularly in thermo-optical properties.

Analysis of results

The analysis of the results may be quite different in the case of a closed analytical solution than for the case of a numerical solution. In the last case, the interpretation of the numerical solution to judge its validity, accuracy and sensitivity to input parameters can be quite involved. The direct solution usually gives just the set of values of the function at the nodes, what is difficult to grasp for humans in raw format (a list of numbers or, for regular meshes, a matrix). Some basic post-processing tools are needed for:

- Visualization of the function by graphic display upon the geometry or at user-selected cuttings. Unfortunately many commercial routines, besides the obvious geometry overlay, only present the function values as a linear sequence of node values and don't allow the user to select cuts. Additional capabilities as contour mapping and pseudo-colour mapping are most welcome.
- Computation of function derivatives (and visualization). Some times only the function is computed, and the user is interested in some special derivatives of the function, as when heat fluxes are needed, besides temperatures.
- Feedback on the meshing, refining it if there are large gradients, or large residues in the overall thermal balance. It is without saying that the user should do all the initial trials (what usually takes the largest share of the effort) with a coarse mesh, to shorten the feedback period.
- Precision and sensitivity analysis by running some trivial cases (e.g. relaxing some boundary condition) and by running 'what-if' type of trials, changing some material property, boundary condition and even the geometry.

A global checking that the detailed solution verifies the global energy equation gives confidence in 'black box' outputs and serves to quantify the order of magnitude of the approximation.

MODELLING HEAT CONDUCTION

MODELLING MASS DIFFUSION

MODELLING HEAT AND MASS CONVECTION

MODELLING THERMAL RADIATION

GENERAL EQUATIONS OF PHYSICO-CHEMICAL PROCESSES

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