Movimiento turbulento bidimensional y estacionario de líquidos (provisional)

1 Ecuaciones

Continuidad

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \tag{1}$$

Cantidad de movimiento

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(-\overline{u'^2}\right) + \frac{\partial}{\partial y}\left(-\overline{u'v'}\right) + \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \tag{2}$$

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(-\overline{u'v'}\right) + \frac{\partial}{\partial y}\left(-\overline{v'^2}\right) + \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right). \tag{3}$$

Energía

$$U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y} = \frac{\partial}{\partial x}\left(-\overline{u'T'}\right) + \frac{\partial}{\partial y}\left(-\overline{v'T'}\right) + \frac{\nu}{Pr}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right). \tag{4}$$

2 Aproximación capa límite

Continuidad

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \tag{5}$$

Cantidad de movimiento

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = U_e\frac{dU_e}{dx} + \frac{\partial}{\partial y}\left(-\overline{u'v'}\right) + \nu\frac{\partial^2 U}{\partial y^2}, \tag{6}$$

$$\frac{\partial}{\partial y}\left(p + \rho\overline{v'^2}\right) = 0\ ;\ \ p + \rho\overline{v'^2} = p_e. \tag{7}$$

Energía

$$U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(-\overline{v'T'}\right) + \frac{\nu}{Pr}\frac{\partial^2 T}{\partial y^2}. \tag{8}$$

3 Turbulencia libre

En este caso $U_e\left(dU_e/dx\right) \approx 0$ y el término viscoso $\nu\left(\partial^2 U/\partial y^2\right) \ll \partial\left(-\overline{u'v'}\right)/\partial y$, de modo que las ecuaciones de continuidad y cantidad de movimiento toman la forma

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0\ ;\ \ U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left(-\overline{u'v'}\right).$$

4 Estela (bidimensional) lejana

$$U = U_\infty + \tilde{U} \ ; \ \tilde{U} << U_\infty \ ; \ v' \sim \tilde{U}.$$

$$\frac{\partial \tilde{U}}{\partial x} + \frac{\partial V}{\partial y} = 0 \ ; \ V \sim \tilde{U}\frac{\delta}{x},$$

$$\left(U_\infty + \tilde{U}\right)\frac{\partial \tilde{U}}{\partial x} + V\frac{\partial \tilde{U}}{\partial y} = \frac{\partial}{\partial y}\left(-\overline{u'v'}\right) \ ; \ U_\infty\frac{\partial \tilde{U}}{\partial x} \sim \frac{U_\infty \tilde{U}}{x} >> \tilde{U}\frac{\partial \tilde{U}}{\partial x} \sim V\frac{\partial \tilde{U}}{\partial y} \sim \frac{\tilde{U}^2}{x},$$

$$U_\infty\frac{\partial \tilde{U}}{\partial x} = \frac{\partial}{\partial y}\left(-\overline{u'v'}\right) \ ; \ \int_{-\infty}^{+\infty}\tilde{U}dy = -\frac{D}{\rho U_\infty} = -I \ ; \ y \to \pm\infty : \ \tilde{U} \to 0, \quad -\overline{u'v'} \to 0.$$

$$\frac{U_\infty}{x} \sim \frac{\tilde{U}}{\delta} \ ; \ \tilde{U}\delta \sim I \quad \Rightarrow \quad \delta \sim \sqrt{\frac{Ix}{U_\infty}} \ ; \ \tilde{U} \sim \sqrt{\frac{IU_\infty}{x}}.$$

Solución autosemejante

$$\delta(x) = a\sqrt{\frac{Ix}{U_\infty}} \ ; \ u_s(x) = b\sqrt{\frac{IU_\infty}{x}} \ ; \ \tilde{U}(x,y) = -u_s(x) f(\eta) \ ; \ -\overline{u'v'} = u_s^2(x) g(\eta) \ ; \ \eta = \frac{y}{\delta(x)},$$

donde a y b son constantes que se han de determinar.

$$U_\infty\frac{\partial \tilde{U}}{\partial x} = \frac{U_\infty u_s}{2x}\left(f + \eta\frac{df}{d\eta}\right) \ ; \ \frac{\partial}{\partial y}\left(-\overline{u'v'}\right) = \frac{u_s^2}{\delta}\frac{dg}{d\eta} \quad \Rightarrow \quad \frac{U_\infty\delta}{2xu_s}\left(f + \eta\frac{df}{d\eta}\right) = \frac{dg}{d\eta},$$

$$\frac{U_\infty\delta}{2xu_s} = \frac{a}{2b} \ ; \quad f + \eta\frac{df}{d\eta} = \frac{2b}{a}\frac{dg}{d\eta}.$$

Modelo de turbulencia

$$-\overline{u'v'} = \nu_T\frac{\partial \tilde{U}}{\partial y} \ ; \quad \nu_T \sim u_s\delta,$$

$$-\overline{u'v'} = -\nu_T\frac{\partial \tilde{U}}{\partial y} = -\frac{\nu_T u_s}{\delta}\frac{df}{d\eta} = u_s^2 g(\eta) \ ; \quad g(\eta) = -\frac{\nu_T}{u_s\delta}\frac{df}{d\eta} = -\frac{1}{R_T}\frac{df}{d\eta}.$$

$$f + \eta\frac{df}{d\eta} = -\left(\frac{2b}{aR_T}\right)\frac{d^2f}{d\eta^2} \ ; \quad \frac{2b}{aR_T} = 1.$$

$$f + \eta\frac{df}{d\eta} + \frac{d^2f}{d\eta^2} = 0 \ ; \quad f(\infty) = f(-\infty) = 0 : \ f = exp\left(-\frac{\eta^2}{2}\right).$$

$$\int_{-\infty}^{+\infty}\tilde{U}dy = -I \quad \Rightarrow \quad \int_{-\infty}^{+\infty} f d\eta = \frac{I}{u_s\delta} = \frac{1}{ab} = \sqrt{2\pi}.$$

Las relaciones

$$ab = 1/\sqrt{2\pi} \quad \text{y} \quad 2b/a = R_T,$$

determinan a y b ya que $R_T \approx 12.5$ es un valor experimental.. El resultado es $a = 0.25$ y $b = 1.58$.

En la estela de un cilindro circular los valores experimentales muestran que $\tilde{U} = -u_s f(\eta)$ para $x > 80$ diámetros y $-\overline{u'v'} = u_s^2 g(\eta)$ para $x > 200$ diámetros.

5 Chorro (bidimensional) lejano

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0\ ;\quad U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left(-\overline{u'v'}\right)\ ;\quad \int_{-\infty}^{\infty}U^2dy=\frac{M}{\rho}=m,$$

ya que

$$\int_{-\infty}^{\infty}\left(U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}\right)dy\equiv\int_{-\infty}^{\infty}\left[\frac{\partial U^2}{\partial x}+\frac{\partial\left(UV\right)}{\partial y}\right]dy=\frac{d}{dx}\int_{-\infty}^{\infty}U^2dy=\int_{-\infty}^{\infty}d\left(-\overline{u'v'}\right)=0.$$

De las ecuaciones se tiene

$$V\sim U\frac{\delta}{x}\ ;\quad \frac{U^2}{x}\sim\frac{\overline{u'^{\,2}}}{\delta}\quad\Rightarrow\quad\frac{\sqrt{\overline{u'^{\,2}}}}{U}\sim\sqrt{\frac{\delta}{x}}\ ;\quad U^2\delta\sim m.$$

Con $\delta\sim x$ se tiene $U_{max}\sim\sqrt{m/x}$. Utilizando la función de corriente

$$U=\frac{\partial\psi}{\partial y}\ ;\quad V=-\frac{\partial\psi}{\partial x},$$

buscamos soluciones de semejanza de la forma

$$\delta=ax\ ;\quad \psi=b\sqrt{mx}F\left(\eta\right)\ ;\quad \left(-\overline{u'v'}\right)=\frac{m}{x}G\left(\eta\right)\ ;\quad \eta=\frac{y}{\delta}.$$

Por lo tanto

$$U=\frac{\partial\psi}{\partial y}=\frac{b}{a}\sqrt{\frac{m}{x}}\frac{dF}{d\eta}\ ;\ V=-\frac{\partial\psi}{\partial x}=b\sqrt{\frac{m}{x}}\left(-\frac{1}{2}F+\eta\frac{dF}{d\eta}\right);$$

$$\frac{\partial U}{\partial x}=-\frac{b}{ax}\sqrt{\frac{m}{x}}\left(\frac{1}{2}\frac{dF}{d\eta}+\eta\frac{d^2F}{d\eta^2}\right)\ ;\quad \frac{\partial U}{\partial y}=\frac{b}{a^2x}\sqrt{\frac{m}{x}}\frac{d^2F}{d\eta^2};$$

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=-\frac{b^2m}{2a^2x^2}\left[\left(\frac{dF}{d\eta}\right)^2+F\frac{d^2F}{d\eta^2}\right]=-\frac{b^2m}{2a^2x^2}\frac{d}{d\eta}\left(F\frac{dF}{d\eta}\right)\ ;\quad \frac{\partial}{\partial y}\left(-\overline{u'v'}\right)=\frac{m}{ax^2}\frac{dG}{d\eta}$$

$$\frac{b^2}{2a}\frac{d}{d\eta}\left(F\frac{dF}{d\eta}\right)=-\frac{dG}{d\eta}$$

Modelo de turbulencia

$$-\overline{u'v'}=\nu_T\frac{\partial U}{\partial y}\ ;\ \nu_T=\frac{u_s\left(x\right)\delta\left(x\right)}{R_T}\ ;\ u_s\left(x\right)=U\left(x,0\right)=\frac{b}{a}\left(\frac{dF}{d\eta}\right)_{\eta=0}\sqrt{\frac{m}{x}},$$

$$-\overline{u'v'}=\nu_T\frac{\partial U}{\partial y}=\frac{b^2F'\left(0\right)m}{a^2R_Tx}\frac{d^2F}{d\eta^2}=\frac{m}{x}G\left(\eta\right)\ ;\ G\left(\eta\right)=\frac{b^2F'\left(0\right)}{a^2R_T}\frac{d^2F}{d\eta^2},$$

$$\frac{d}{d\eta}\left(F\frac{dF}{d\eta}+\frac{d^2F}{d\eta^2}\right)=0\quad\text{con}\quad\frac{2F'\left(0\right)}{aR_T}=1\quad\text{y con}\quad F\left(0\right)=F''\left(0\right)=0\quad\text{y}\quad F'\left(\infty\right)=0.$$

Integrando una vez se tiene

$$F\frac{dF}{d\eta}+\frac{d^2F}{d\eta^2}=0,$$

llegándose a

$$F\left(\eta\right)=\sqrt{2}tanh\left(\eta/\sqrt{2}\right)\ ;\ F'\left(\eta\right)=sech^2\left(\eta/\sqrt{2}\right)=\frac{4}{\left(e^{\eta/\sqrt{2}}+e^{-\eta/\sqrt{2}}\right)^2}.$$

Dado que $F'(0) = 1$ se tiene $a = 2/R_T \approx 0.078$ ($R_T \approx 25.7$). Por otro lado se tiene

$$\int_{-\infty}^{\infty} U^2 dy = m \quad \Rightarrow \quad \int_{-\infty}^{\infty} \left(\frac{dF}{d\eta}\right)^2 d\eta = \frac{a}{b^2},$$

pero, dado que $F'(\eta)$ es conocida, se tiene

$$\int_{-\infty}^{\infty} \left(\frac{dF}{d\eta}\right)^2 d\eta = \frac{4\sqrt{2}}{3} = \frac{a}{b^2} \quad \Rightarrow \quad b \approx 0.203.$$

En resumen se tiene

$$\delta(x) \approx 0.078x \ ; \ \ u_s(x) = U(x,0) \approx 2.60\sqrt{\frac{m}{x}} \ ; \ \ \psi \approx 0.203\sqrt{mx}F(\eta) \ ; \ \ \eta = \frac{y}{\delta(x)}.$$